Strategic Sensors and Regional Boundary Exponential Observation

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Abstract:

In this paper an extension of regional exponential observability concept to the case of boundary region of $\partial\Omega$ has been discussed and analyzed in connection with the strategic sensors. For distributed parameter systems of parabolic type, we show that, the number and location of sensor may be some interest in the existence of regional exponential observation state in this region.

Keywords: Γ- strategic sensor, Γ-exponential detectability, Γ-exponential observability.

1. Introduction

In modern mathematical control theory, observability means that it is possible to reconstruct uniquely the initial state of the dynamic system from the knowledge of the input and output [1-2]. Notion of regional observability (extended by El Jai *et al.* [3-4]) is of great importance in current research and motivated by many applications [4-7]. The concept of regional asymptotic analysis was explored by Al-Saphory and El Jai [8-10], and this concept consist to study the behavior of the system not in whole domain Ω but only on particular region ω of the domain Ω. The purpose of this paper, is to bring the light to link between the regional boundary exponential observability and sensor structure (location and number (see figure 1).

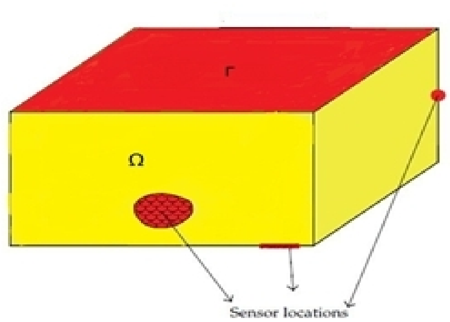

Figure1. The domain Ω, the boundary region Γ, and the sensors locations

We consider a class of distributed system and develop different results connected with the various types of measurements, and we define anew type of strategic sensor which maybe regional boundary exponential strategic sensor.

This paper is organized as follows: section one is focused on preliminaries and the problem formulation. In the next section, the characterization notion of regional boundary exponential observability is given by using of strategic sensors. The last section is devoted to applications with many situations of sensor locations.

2. Regional boundary exponential observability

In this section we give firstly, the statement of the problem with the hypothesis of considered system, and then the concept of regional boundary exponential observability is explained, and we provide a theorem which gives the approach observed the current state $z(\mu, t)$of the original system (1) exponentially.

2.1 Preliminaries

The considered system is described by the parabolic equations;

$$\begin{cases} \frac{\partial z}{\partial t}(\mu, t) = Az(\mu, t) + Bu(t) & \Omega \times (0, \infty) \\ z(\mu, 0) = z_0(\mu) & \bar{\Omega} \\ \frac{\partial z}{\partial \nu_A}(\mu, t) = 0 & \partial\Omega \times (0, \infty) \end{cases} \tag{1}$$

Where Ω is the domain when the above system is defined as bounded open subset of R^n with boundary $\partial\Omega$, $[0,T]$ is the time interval for $T > 0$, A is a second order linear differential operator and is self adjoint with compact resolvant , and which generates astrongly continuous semi-group $(S_A(t))_{t\geq 0}$ on the state space $Z = H^1(\Omega)$ which is Sobolev space of order one, A^* will denote the adjoint operator of A. The operators $B \in L(R^p, Z)$ and $C \in L^2(0, T, R^q)$, where p and q is the number of actuators and sensors. The initial state $z_0(\mu)$ is supposed to be unknown and located in $H^1(\bar{\Omega})$. The measurements of system (1) are obtained through internal or boundary zone or pointwise sensors which characterize the output function

$$y(.,t) = Cz(\mu, t) \tag{2}$$

Under the above assumption, the system (1) has a unique solution [1-2]

$$z(\mu, t) = S_A(t)z_0(\mu) + \int_0^t S_A(t-s)Bu(s)ds \tag{3}$$

Now, we define the following operators

$$\begin{aligned} K &: Z \to \mathcal{O} \\ z &\to CS_A(.)z \end{aligned} \tag{4}$$

with an adjoint

$$K^{*}: \mathcal{O} \to Z \tag{5}$$

given by

$$K^{*}y^{*} = \int_0^t S_A{}^{*}(s)C^{*}y^{*}(s)ds \tag{6}$$

We also consider the trace operator of order zero

$$\gamma_0: H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega) \tag{7}$$

Which is linear, subjective, and continuous, such that z_0^{Γ} is the restriction of the trace of the initial state z_0 to Γ.

$\gamma_0{}^{*}$ denote the adjoint of $\gamma_0$given by

$$\gamma_0{}^{*}: H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \tag{8}$$

and

$$\chi_\Gamma: H^{\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\Gamma) \tag{9}$$

$\chi_\Gamma^{\bar{*}}$ is the operator restriction to Γwhich is the adjoint of χ_Γ given by

$$\chi_\Gamma^{\bar{*}}: H^{\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\partial\Omega) \tag{10}$$

Definition 2.1. The autonomous system associated with system (1) together with output function (2) is said to be exactly regionally boundary observable (or exactly Γ- observable) if

$$\mathrm{Im}(\chi_\Gamma \gamma_0 K^{*}) = H^{\frac{1}{2}}(\Gamma) \tag{11}$$

Definition 2.2. The autonomous system associated with system (1) together with output function (2) is said to be weakly regionally boundary observable (or weakly Γ- observable) if

$$\overline{\mathrm{Im}(\chi_{\Gamma}\gamma_0 K^*)} = H^{\frac{1}{2}}(\Gamma) \tag{12}$$

2.2 Γ- strategic sensor

In this subsection we give a characterization of strategic sensor as in ref. [12] in order the system to be weakly Γ- observable.

Definition 2.3. Sensors are any couple $(D_i, f_i)_{1\le i\le q}$ where D_i form closed subsets of $\overline{\Omega}$, which are spatial supports of sensors and $f_i \in L^2(D_i)$ define the spatial distribution of sensors on D_i.

In case, when the measurements of system (1) are given by ***i*** sensors $(\boldsymbol{1 \le i \le q})$,Thenthe output function (2) is given by;

$$y(.,t) = y_1(.\ ,t), \dots\ , y_q(.\ ,t) \tag{13}$$

with

$$y_i(.,t) = z(b_i\ ,t), b_i \in \overline{\Omega}\ for\ 1 \le i \le q \tag{14}$$

in the pointwise case, and

$$y_i(.,t) = \int_{D_i} z(\mu, t) f_i(\mu) d\mu, D_i \in \overline{\Omega} for\ 1 \le i \le q \tag{15}$$

in the zonal case.

Definition 2.4. A sequence of sensors $(D_i, f_i)_{1\le i\le q}$ is called boundary strategic on Γ (or Γ – strategic) if the corresponding system is weakly Γ- observable.

Now, assume that there exist a complete set ofeigenfunctions $(\varphi_{mj})_{m\in I}$, j = 1, … , $r_m$of A in state space $H^1(\Omega)$ associated with the eigenvalues λ_m of multiplicities r_m. Let $r = \sup_{m\in I} r_m$ is finite and for $z = (z_1, \dots, z_n) \in \Omega$ with $m = (m_1, \dots, m_n) \in I$, suppose that $\bar{z} = (z_1, \dots, z_{n-1})$ and $\overline{m} = (m_1, \dots, m_{n-1})$.

Suppose that the functions $\psi_{\bar{m}j}(\bar{z}) = \chi_\Gamma \gamma_0 \varphi_{mj}(z)$, $m \in I$, form a complete set in $H^{\frac{1}{2}}(\Gamma)$, Then we have the following result.

Proposition 2.5. The sequence of sensors $(D_i, f_i)_{1 \le i \le q}$ is Γ- strategic if and only if:

1. $q \ge r$
2. rank $G_m = r_m, \quad \forall m, \; m = 1, \dots, J$ with

$$G_m = (G_m)_{ij} = \begin{cases} < \varphi_{mj}, f_i(.) >_{L^2(D_I)} & \text{in the zone case} \\ \varphi_{mj}(b_i) & \text{in the pointwise case} \end{cases} \tag{16}$$

where $\sup r_m = r$ and $J = 1, \dots, r_m$.

Proof: The proof of this proposition is similar to the rank condition in [13-15], the main difference is that the rank condition

rank $G_m = r_m, \quad \forall m$

For the proposition 2.1.need only hold for rank $G_m = r_m, \quad \forall m, \; m = 1, \dots, J$.■

2.3 Γ_E- Observability

In this subsection an interested extension of regional case as in ref. [12] is developed to the boundary case. Thus, the characterization of regional boundary exponential observation needs some notions which are related to the exponential behavior that are stability, detectability, and observer. In this subsection we define the concepts which are related to Γ_E- Observability, and we provide an important theorem which gives an exponential observer for the original system in critical subregion Γ.

The regional boundary exponential observer in Γ may be seen as internal regional exponential observer in ω if we consider the following.

- Let $\Re$ be the continuous liner extension operator [16]

  $\Re : H^{\frac{1}{2}}(\partial\Omega) \longrightarrow H^1(\Omega)$ such that

$$\gamma_0 \Re h(\mu, t) = h(\mu, t), \quad \forall h \in H^{\frac{1}{2}}(\partial\Omega) \tag{17}$$

- Let $r > 0$ is an arbitrary and sufficiently small real and let the sets
  $E = \bigcup_{z\in\Gamma} B(z,r)$ and $\omega_r = E \cap \Omega$ (18)

where $B(z,r)$ is the ball of radius *r* centered in $z(\mu,t)$ and Γ is a part of $\bar{\omega}_r$(fig.2).

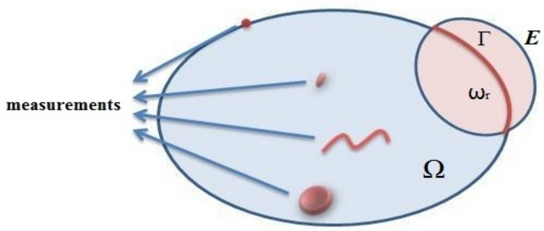

Fig. 2: The whole domain Ω and the set *E*.

Definition 2.6. The semi-group $(S_A(t))t \geq 0$ is regionally boundary exponentially stable *in* $H^{\frac{1}{2}}(\Gamma)$(or Γ_E-stable)if, for some positive constants $F_\Gamma$and σ_Γ, then

$$\|\chi_\Gamma\gamma_0 S_A(.)\|_{H^{\frac{1}{2}}(\Gamma)} \leq F_\Gamma e^{-\sigma_\Gamma t}, t \geq 0 \tag{19}$$

If $(S_A(t))_{t\geq 0}$ is Γ_E-stable, then for every $z_0(.)$ □ $H^1(\bar{\Omega})$ the solution of autonomous system associated with (1) satisfies

$$\lim_{t\to\infty}\|z(t)\|_{H^{\frac{1}{2}}(\Gamma)} = \lim_{t\to\infty}\|\chi_\Gamma\gamma_0 S_A(t)z_0\|_{H^{\frac{1}{2}}(\Gamma)} = 0 \tag{20}$$

Definition 2.7. The system (1) is regionally boundary exponentially stable on Γ (orΓ_E-stable)if, the operator A generates a strongly continuous semi-group $(S_A(t))_{t\geq 0}$ which is Γ_E-stable.

Remark 2.8. If the system (1) is Γ_E-stable, that is the solution of autonomous system associated with (1) converges exponentially to zero when ***t*** tends to ∞.

Definition 2.9. The system (1) - (2) is regionally boundary exponentially detectable on Γ (or Γ_E- detectable) if there exist an operator $H_\Gamma: \mathcal{O} \to H^{\frac{1}{2}}(\Gamma)$ such that the operator $A - H_\Gamma C$ generates a strongly continuous semigroup $(S_{H_\Gamma}(t))_{t\geq 0}$ which is Γ_E-stable.

Now**,** consider the dynamical system

$$\begin{aligned}
&\frac{\partial \boldsymbol{w}}{\partial \boldsymbol{t}}(\boldsymbol{\mu},\boldsymbol{t}) = \boldsymbol{L}_\Gamma \boldsymbol{w}(\boldsymbol{\mu},\boldsymbol{t}) + \boldsymbol{G}_\Gamma \boldsymbol{u}(\boldsymbol{t}) + H_\Gamma \boldsymbol{y}(\boldsymbol{t}) && \boldsymbol{\Omega}\times(\boldsymbol{0},\infty)\\
&\boldsymbol{w}(\boldsymbol{\mu},\boldsymbol{0}) = \boldsymbol{w_0}(\boldsymbol{\mu}) && \overline{\boldsymbol{\Omega}} \qquad (19)\\
&\frac{\partial \boldsymbol{w}}{\partial \boldsymbol{\nu}_{\boldsymbol{L}_\Gamma}}(\boldsymbol{\mu},\boldsymbol{t}) = \boldsymbol{0} && \boldsymbol{\partial\Omega}\times(\boldsymbol{0},\infty)
\end{aligned}$$

where $\boldsymbol{L}_\Gamma$ generates a strongly continuous semi-group $(S_{L_\Gamma}(t))_{t\geq 0}$ which is stable on the state space W, G_Γ □ $L(R^p,W)$ and H_Γ □ $L(R^q,W)$. The system (13) defines an Γ_E-estimater for $\alpha_\Gamma z(\mu,t)$ if

(1) $\lim_{t\to\infty}\|w(.,t)-\alpha_\Gamma z(.,t)\|_{H^{\frac{1}{2}}(\Gamma)} = 0$ (20)

(2) $\alpha_\Gamma$maps D(A) into D(L_Γ) where $\alpha_\Gamma = \chi_\Gamma\gamma_0 T$ and w(.,t) is the solution of system(19).

Definition 2.10.The system (19) is regionally boundary exponential observer on Γ (or Γ_E-observer) for the system (1)-(2) if the following conditions hold

(1) The exist operators M_Γ □ $L\left(\mathcal{O},H^{\frac{1}{2}}(\Gamma)\right)$ *and* N_Γ □ $L\left(H^{\frac{1}{2}}(\Gamma)\right)$ such that:

$M_\Gamma C + N_\Gamma\alpha_\Gamma = I_\Gamma$(21)

$\alpha_\Gamma A + L_\Gamma\alpha_\Gamma = H_\Gamma C \text{ and } G_\Gamma = \alpha_\Gamma B$

(2) The system (19) defines an Γ_E-estimator for the state of original system.

Definition 2.11. The system (19) is regionally boundary exponential identity observer (or identity Γ_E-observer) for the system (1)-(2) if Z = W and $\alpha_\Gamma = I_\Gamma$. In the this case, we have

$$L_\Gamma = A - H_\Gamma C \text{ and } G_\Gamma = B \qquad (22)$$

and then, the dynamical system (19) becomes

$$\begin{aligned}
&\frac{\partial \boldsymbol{w}}{\partial \boldsymbol{t}}(\boldsymbol{\mu},\boldsymbol{t}) = \boldsymbol{A}\boldsymbol{w}(\boldsymbol{\mu},\boldsymbol{t}) + \boldsymbol{B}\boldsymbol{u}(\boldsymbol{t}) + H_\Gamma(\boldsymbol{C}\boldsymbol{w}(\boldsymbol{\mu},\boldsymbol{t}) - \boldsymbol{y}(\boldsymbol{\mu},\boldsymbol{t})) && \boldsymbol{\Omega}\times(\boldsymbol{0},\infty)\\
&\boldsymbol{w}(\boldsymbol{\mu},\boldsymbol{0}) = \boldsymbol{0} && \overline{\boldsymbol{\Omega}} \qquad (23)\\
&\frac{\partial \boldsymbol{w}}{\partial \boldsymbol{\nu}_{\boldsymbol{A}}}(\boldsymbol{\mu},\boldsymbol{t}) = \boldsymbol{0} && \boldsymbol{\partial\Omega}\times(\boldsymbol{0},\infty)
\end{aligned}$$

Definition 2.12. We say that the system (1)-(2) isregionally boundary exponentially observable on Γ (or Γ_E-observable) if, there exist a dynamical system (19) which is Γ_E-observer for the original system.

In the following, we present an approach which is observed the current state $z(\mu, t)$of the original system (1) exponentially.

3. Γ_E-observability and strategic sensor

The problem of Γ_E-observability is consist of estimate the current state exponentially in a given boundary sub-region Γ of the boundary $\partial\Omega$, this approach is given by the following main result.

Theorem 3.1. *Suppose that the sequence of sensors $(D_i, f_i)_{1\le i\le q}$ is Γ-strategic and the spectrum of A contain J eigenvalues with non- negative real parts. Then (1)-(2) is Γ_E- observable by the dynamical system*

$$\begin{aligned} &\frac{\partial w}{\partial t}(\mu, t) = Aw(\mu, t) + Bu(t) + H_\Gamma(Cw(\mu, t) - y(\mu, t)) && \Omega \times (0, \infty) \\ &w(\mu, 0) = w_0(\mu) && \bar{\Omega} \qquad (24) \\ &\frac{\partial w}{\partial \nu_A}(\mu, t) = 0 && \partial\Omega \times (0, \infty) \end{aligned}$$

Proof: The demonstration has two parts:

Part1. Under the hypothesis of problem (Section 2.1), the system (1) can be decomposed by the projections P and $I - P$ on two parts, unstable and stable. The state vector may be given by

$$z(\mu, t) = [z_1(\mu, t) + z_2(\mu, t)]^{tr} \qquad (25)$$

where $z_1(\mu, t)$ is the state component of the unstable part of thesystem (1) and may be written in the form

$$\frac{\partial z_1}{\partial t}(\mu,t) = A_1 z_1(\mu,t) + PBu(t)\Omega \times (0,\infty)$$

$$z_1(\mu,0) \quad = z_0^1(\mu) \qquad \bar{\Omega} \qquad (26)$$

$$\frac{\partial z_1}{\partial \nu_A}(\mu,t) = 0 \qquad \partial\Omega \times (0,\infty)$$

and $z_2(\mu,t)$ is the state component of the stable part of (1) given by

$$\frac{\partial z_2}{\partial t}(\mu,t) = A_2 z_2(\mu,t) + (I-P)Bu(t) \qquad \Omega \times (0,\infty)$$

$$z_2(\mu,0) = z_0^2(\mu) \qquad \bar{\Omega} \qquad (27)$$

$$\frac{\partial z_2}{\partial \nu_A}(\mu,t) = 0 \qquad \partial\Omega \times (0,\infty)$$

The operator A_1 is represented by matrix of order $(\sum_{n=1}^{J} r_n, \sum_{n=1}^{J} r_n)$ given by

$$A_1 = diag\lfloor \lambda_1, \dots, \lambda_1, \lambda_2, \dots, \lambda_2, \lambda_j, \dots, \lambda_j \rfloor,$$

$$PB = [G_1^{tr}, G_2^{tr}, \dots, G_J^{tr}] \qquad (28)$$

Part 2. Since the suite of sensors $(D_i, f_i)_{1\le i\le q}$ is Γ-strategic for the unstable part of the system(1), then the subsystem (27) is weakly Γ-observable [14], and since it is finite dimensional, then it is exactly Γ-observable therefore, it is $\Gamma_E$detectable [10], and hence there exists an operator H_Γ^1such that $A_1 - H_\Gamma^1 C$satisfies the following:

 $F_\Gamma^1$and$\sigma_\Gamma^1 > 0$ such that $\left\| e^{(A_1 - H_\Gamma^1 C)t} \right\| \le F_\Gamma^1 e^{-\sigma_\Gamma^1 t}$and, then, we have

$$\| z_1(\mu,t) \|_{H^{\frac{1}{2}}(\Gamma)} \le F_\Gamma^1 e^{-\sigma_\Gamma^1 t} \| P z_0^1 \|_{H^{\frac{1}{2}}(\Gamma)} \qquad (29)$$

Now, since A_2 generates semigroup which is Γ_E-stable, then, there is appositive constants F_Γ^2 *and* σ_Γ^2 such that

$$\boldsymbol{\| z_2(\mu,t) \|_{H^{\frac{1}{2}}(\Gamma)} \le F_\Gamma^2 e^{-\sigma_\Gamma^2 t} \left\| (I-P) z_0^2 \right\|_{H^{\frac{1}{2}}(\Gamma)} + \int_0^t F_\Gamma^2 e^{-\sigma_\Gamma^2 t} \left\| (I-P) z_0^2 \right\|_{H^{\frac{1}{2}}(\Gamma)} \| Bu(s) \|_{H^{\frac{1}{2}}(\Gamma)} ds} \qquad (30)$$

and therefore $\| z(\mu,t) \|_{H^{\frac{1}{2}}(\Gamma)}$ converges to zero when *t* tends to ∞**.** Finally, the system (1)-(2) is Γ_E detectable.

Now, Let $e(\mu,t) = z(\mu,t) - w(\mu,t)$where $z(\mu,t)$is solution of the system(1)-(2)and $w(\mu,t)$ is the solution of (24). Driving the above equation and using (1) and (24), we obtain

$$\begin{aligned}\frac{\partial e}{\partial t}(\mu,t) &= \frac{\partial z}{\partial t}(\mu,t) - \frac{\partial w}{\partial t}(\mu,t)\\ &= Az(\mu,t) + Bu(t) - Aw(\mu,t) - Bu(t) - H_\Gamma(Cw(\mu,t) - y(\mu,t))\\ &= A\big(z(\mu,t) - w(\mu,t)\big) - H_\Gamma C(w(\mu,t) - z(\mu,t))\\ &= (A - H_\Gamma C)\, e(\mu,t) \qquad (31)\end{aligned}$$

Since the system (1)-(2) is Γ_E detectable, there exists an operator H_Γ □ $L(\mathcal{O}, H^{\frac{1}{2}}(\Gamma))$ such that the operator $A - H_\Gamma C$ generates Γ_E-stable semigroup $(S_{H_\Gamma}(t))_{t\geq 0}$ satisfies the following relation:

□ F_Γ *and* $\sigma_\Gamma > 0$ such that

$$\left\|\chi_\Gamma \gamma_0 S_{H_\Gamma}(t)\right\|_{H^{\frac{1}{2}}(\Gamma)} \leq F_\Gamma e^{-\sigma_\Gamma t} \qquad (32)$$

Consequently, we get

$$\begin{aligned}\|e(\mu,t)\|_{H^{\frac{1}{2}}(\Gamma)} &= \|z(\mu,t) - w(\mu,t)\|_{H^{\frac{1}{2}}(\Gamma)}\\ &\leq \left\|\chi_\Gamma \gamma_0 S_{H_\Gamma}(t) e_0\right\|_{H^{\frac{1}{2}}(\Gamma)}\\ &\leq F_\Gamma e^{-\sigma_\Gamma t}\|e_0\|_{H^{\frac{1}{2}}(\Gamma)} \qquad (33)\end{aligned}$$

that is $e(\mu,t)$ converges exponentially to zero as t tends to ∞. Thus, the dynamical system (24) observes exponentially the regional state $z(\mu,t)$ of the original system (1)-(2), and therefore it is Γ_E-observable.■

Now, we define a new type of strategic sensors.

Definition3.2. The suite of sensor $(D_i, f_i)_{1\leq i\leq q}$ is said to be regional boundary exponential strategic (or Γ_E-strategic) if, the corresponding system is Γ_E- observable.

Remark 3.3. We can deduce that:

(1)A system which is exponentially observable (on Ω) is Γ_E-observable,

(2) A system which is exactly Γ -observable is Γ_E-observable,

(3)A system which is Γ_E-observable is Γ_E^1-observable, for every subset Γ^1 subset of Γ.

Remark 3.3. One important point in this subsection is that a state which is not exponentially observable in the usual sense may be exponentially observable on Γ, this is illustrated by the following counter example.

Example 3.4. (*counter example***)**

Consider the system described by the following diffusion equation

$$\begin{cases} \frac{\partial z}{\partial t}(\mu_1,\mu_2,t) = \frac{\partial^2 z}{\partial \mu_1^2}(\mu_1,\mu_2,t) + \frac{\partial^2 z}{\partial \mu_2^2}(\mu_1,\mu_2,t) + z(\mu_1,\mu_2,t) & \Omega \times (0,T) \\ z(\mu_1,\mu_2,0) = z_0(\mu_1,\mu_2) & \bar{\Omega} \\ \frac{\partial z}{\partial \nu}(\mu_1,\mu_2,t) = 0 & \partial\Omega \times (0,T) \end{cases} \tag{34}$$

With output

$$y(t) = \int_\Omega z(\mu_1,\mu_2,t)\mathcal{S}((\mu_1 - b_1,\mu_2 - b_2)d\mu_1 d\mu_2 \tag{35}$$

Where $\Omega = (0,\alpha) \times (0,\beta)$ and $b = (b_1,b_2)$ □ Ω is the location of sensor $(b,\mathcal{S}_b)$ (as in figure 2). The operator $A = (\frac{\partial^2}{\partial \mu_1^2} + \frac{\partial^2}{\partial \mu_2^2} + 1)$ generates a strongly continuous semigroup on the state space $H^1(\Omega)$.

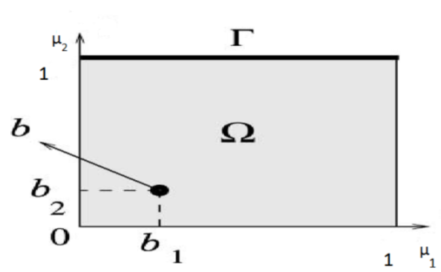

Figure 2: Rectangular domain, region $\Gamma$and location ***b***of internal pointwise sensor.

Now, we assume that $\Omega \times (0, \infty) = \mathcal{Q}, \partial\Omega \times (0, \infty) = \Sigma$ and consider the dynamical system

$$
\begin{aligned}
&\frac{\partial w}{\partial t}(\mu_1,\mu_2,t) = \frac{\partial^2 w}{\partial \mu_1^2}(\mu_1,\mu_2,t) + \frac{\partial^2 w}{\partial \mu_2^2}(\mu_1,\mu_2,t) + w(\mu_1,\mu_2,t) + \mathrm{HC}(w(\mu_1,\mu_2,t) - z(\mu_1,\mu_2,t) && \mathcal{Q} \\
&w(\mu_1,\mu_2,0) = w_0(\mu_1,\mu_2) && \overline{\Omega} \qquad (36) \\
&\frac{\partial w}{\partial \nu}(\mu_1,\mu_2,t) = 0 && \Sigma
\end{aligned}
$$

where $H \in L(\mathcal{O}, W)$, W is the state space of above system. Here if $\{\frac{b_1}{\alpha}, \frac{b_2}{\beta}\} \in \mathbb{Q}$, then the sensor $(b, \mathcal{S}_b)$ is not strategic [1] for the unstable subsystem (34), and therefore the system (34)-(35) is not exponentially detectable in Ω, and then, the dynamical system (36) is not exponentially observer for (34)-(35), finally the state $z_0(\mu_1,\mu_2)$ is not exponentially observable in Ω.

Now, we consider the boundary region $\Gamma = (0, \alpha) \times \{\beta\} \subset \partial\Omega$, (figure .2) and the dynamical system

$$
\begin{aligned}
&\frac{\partial w}{\partial t}(\mu_1,\mu_2,t) = \frac{\partial^2 w}{\partial \mu_1^2}(\mu_1,\mu_2,t) + \frac{\partial^2 w}{\partial \mu_2^2}(\mu_1,\mu_2,t) + w(\mu_1,\mu_2,t) + \mathrm{H}_\Gamma \mathrm{C}(w(\mu_1,\mu_2,t) - z(\mu_1,\mu_2,t)) && \mathcal{Q} \\
&w(\mu_1,\mu_2,0) = w_0(\mu_1,\mu_2) && \overline{\Omega} \qquad (37) \\
&\frac{\partial w}{\partial \nu}(\mu_1,\mu_2,t) = 0 && \Sigma
\end{aligned}
$$

where $\mathrm{H}_\Gamma \in L(\mathcal{O}, H^{\frac{1}{2}}(\Gamma)$. If there exist $i, j \in \{1, \dots, J\}$such that $\frac{ib_1}{\alpha}$ *and* $\frac{ib_2}{\beta} \in$ Ithen, the sensor $(b, \mathcal{S}_b)$ is Γ-strategic [7], and this implies to the system (34)-(35) is Γ-detectable, therefore the may be Γ_E-observable.■

4. Relations between Γ_E-observability and the sensors position

In this section, we apply the previous results to a two dimensional system defined on rectangular domain once and on disk domain twice.

4.1 Rectangular domain

Consider the diffusion system

$$\begin{cases} \dfrac{\partial z}{\partial t}(\mu_1,\mu_2,t) = \Delta z(\mu_1,\mu_2,t) + z(\mu_1,\mu_2,t) + Bu(t) & Q \\ z(\mu_1,\mu_2,0) = z_0(\mu_1,\mu_2) & \bar{\Omega} \\ \dfrac{\partial z}{\partial v}(\mu_1,\mu_2,t) = 0 & \Sigma \end{cases} \tag{38}$$

with measurements obtained by output function given as (2), where $\Omega = (0,1) \times (0,1)$, $\Gamma = (0,1) \times \{1\}$, in this case the eigenfunctions of the dynamic system (38) for Derichlet boundary conditions are given by:

$$\varphi_{ij}(\mu_1,\mu_2) = 2\cos(i\pi\mu_1)\cos(i\pi\mu_2) \tag{39}$$

associated with eigenvalues

$$\lambda_{ij} = -(i^2 + j^2)\pi^2 \tag{40}$$

The following results give information on the location of internal zone or pointwise Γ -strategic sensors.

4.1.1 **Internal Pointwise Sensor**

Let us consider the case of pointwise sensor located inside of $\mathbf{\Omega}$. The system (38) is augmented with the following output function:

$$y(t) = \int_{\Omega} z(\mu_1,\mu_2,t)\, \mathcal{S}(\mu_1 - b_1, \mu_2 - b_2) d\mu_1 d\mu_2, \tag{41}$$

such that the sensor $\mathrm{b} = (b_1, b_2)$ □ Ω**,** (see figure 3).

If there exist i, j □ $\{1, \dots, J\}$, such that ib_1, ib_2 □ $I, r_m = 1$, then the sensor $b = (b_1, b_2)$ may be sufficient for Γ_E-observability, with the dynamical system

$$\begin{cases} \dfrac{\partial w}{\partial t}(\mu_1,\mu_2,t) = \Delta w(\mu_1,\mu_2,t) + w(\mu_1,\mu_2,t) + Bu(t) + H_\Gamma(w(\mu_1,\mu_2,t) - y(t) & Q \\ w(\mu_1,\mu_2,0) = w_0(\mu_1,\mu_2) & \bar{\Omega} \\ \dfrac{\partial w}{\partial v}(\mu_1,\mu_2,t) = 0 & \Sigma \end{cases} \tag{42}$$

Forms an Γ_E-observer for (38), thus we obtain the following result

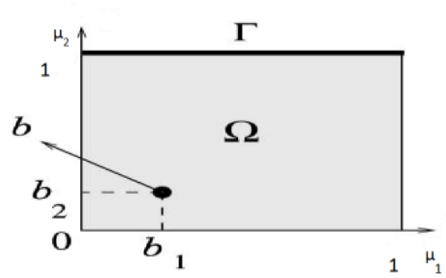

Figure 3: Rectangular domain, region Γ and location **b** of internal pointwise sensor.

Corollary 4.1. *The system (38)-(41) is Γ_E-observable by the dynamical system (42), If there exist $i, j \in \{1, \dots, J\}$, such that $ib_1, jb_2 \in I$.*

4.2. Internal Zone Sensor

Consider the system (38) together with output function (2) where the sensor support D is located in Ω. The output function (2) can be written by the form

$$y(t) = \int_{\mathrm{D}} z(\mu_1, \mu_2, t)\, f(\mu_1, \mu_2) d\mu_1 d\mu_2, \tag{43}$$

where $D = [\mu_{1_0} - l_1, \mu_{1_0} + l_1] \times [\mu_{2_0} - l_2, \mu_{2_0} + l_2] \subset \Omega$ is the location of sensor (D, f) *and* $f \in L^2(D)$ (see fig. 4), if f is not symmetric with respect to $\mu_k = \mu_{k_0}, k = 1,2$, there exist $i, j \in \{1, \dots, J\}$, such that $i\mu_{1_0}, j\mu_{2_0} \in I$ and $r_m = 1$, then the sensor (D, f) may be sufficient for Γ_E-observability, with the dynamical system:

$$\begin{array}{ll} \frac{\partial w}{\partial t}(\mu_1, \mu_2, t) = \Delta w(\mu_1, \mu_2, t) + w(\mu_1, \mu_2, t) + Bu(t) + H_\Gamma(\langle w(\mu_1, \mu_2, t), f(\mu_1, \mu_2)\rangle - y(t)) & Q \\ w(\mu_1, \mu_2, 0) = w_0(\mu_1, \mu_2) & \bar{\Omega} \\ \frac{\partial w}{\partial \nu}(\mu_1, \mu_2, t) = 0 & \Sigma \end{array} \tag{44}$$

thus we obtain the following result:

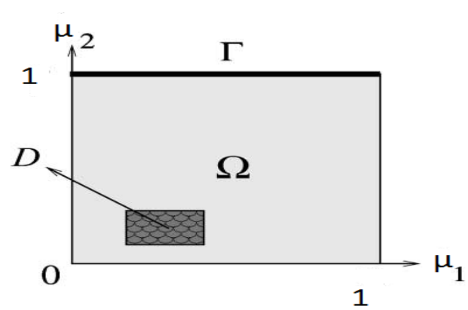

Figure 4: Rectangular domain, region Γ and location D of internal zone sensor.

Corollary 4.2.*The system (38)-(43) is Γ_E-observable by the dynamical system (44), If there exist $i, j \in \{1, \dots, J\}$, such that $i\mu_{1_0}, j\mu_{2_0} \in I$and f isnot symmetric with respect to* $\mu_k = \mu_{k_0}, k = 1,2$.

4.3. Internal Filament Sensor

Consider the case of the observation on the curve $\sigma = Im(\gamma)$ with $\gamma \in C^1(0,1)$ (see figure 5),then we have the following.

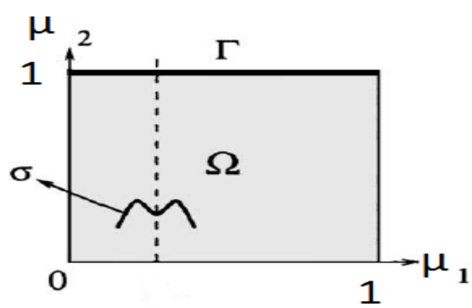

Figure 5: Rectangular domain, region Γ and location σ of internal filament sensor.

Corollary 4.3. If theobservation recovered by filament sensor $(\sigma, \delta\sigma)$ such that it is symmetric with respect to the line $\mu = \mu_0$, then the system (38) with output given by

$$y(t) = \int_\Omega z(\mu i_1, \mu i_2, t)\, \mathcal{S}(\mu i_1 - b i_1, \mu i_2 - b i_2) d\mu i_1 d\mu i_2, i = 1, \dots, q \qquad (45)$$

Is not *Γ_E-observable if* there exist $i, j \in \{1, \dots, J\}$, such that$i\mu_{1_0}, j\mu_{2_0} \in I$.

4.2 Disc domain: In this case, we consider the system

$$\begin{cases} \frac{\partial z}{\partial t}(r,\theta,t) = \Delta z(r,\theta,t) + z(r,\theta,t) + Bu(t) & Q \\ z(r,\theta,0) = z_0(r,\theta) & \bar{\Omega} \\ \frac{\partial z}{\partial \nu}(r,\theta,t) = 0 & \Sigma \end{cases} \qquad (46)$$

where $\Omega = D(0,1)$, $\theta \in [0,2\pi]$,$\Gamma = D(1,\theta_i)_{2 \le i \le q}$.

Remark 4.4. We can extended this case into different type of sensors (internal or boundary, zonal or pointwise) as in [10].

5. Conclusion

The concept developed in this paper is related to the regional boundary exponential observability in connection with the strategic sensors. It permits us to avoid some “bad” sensor locations. Various interesting results concerning the choice of sensors structure are given and illustrated in specific situations. Many questions still opened. This is the case of, for example, the problem of finding the optimal sensor location ensuring such an objective.

References

[1] A. El Jai and A. J. Pritchard, Sensors and actuators in distributed systems, *International Journal of Control,* vol. 46, no. 4, pp. 1139–1153, **1987.**

[2] R. F. Curtain and H. Zwart, An introduction to Infinite-Dimensional Linear Systems Theory, *Springer*, New York, NY, USA, **1995.**

[3] A. El Jai, M. C. Simon, and E. Zerrik, Regional observability and sensor structures,*Sensors and Actuators A*, vol. 39, no. 2, pp. 95–102, **1993.**

[4] A. El Jai, M. C. Simon, E. Zerrik, and M. Amouroux,Regional observability of a thermal process,*IEEE Transactions on Automatic Control*, vol. 40, no. 3, pp. 518–521, **1995.**

[5] A. El Jai, Guisset, E.Trombe, and A. Suleiman, Application of boundary observation to a thermal systems. *Conference of MTNS* 2000, Perpignan, France, June 19-23, **2000.**

[6] Y. Quan,and A. Moore, Diffusion boundary determination and zone control via ,mobile actuators-sensors networks (MAS-net)-challenge and opportunities, *SCOIS, Utah state university, Logan, UT84322, USA,* 20**04.**

[7] J. Burns, J. Jorggaard, M. Cliff, and L. Zeietman, A PDE approach to optimization and control of high performance buildings, *workshop in Numerical Techniques for Optimization Problems with PDE Constraints, No. 04/2009, DOI: 10.4171/OWR/2009/04,* 25-31 January, Houston, Denmark, 20**09.**

[8] R. Al-Saphory, “Sensor structures and regional exponential observability,”International Scholarly Research NetworkISRN Applied Mathematics,Volume 2011, Article ID 673052, doi:10.5402/2011/673052, 14 pages, **2011.**

[9] R. Al-Saphory and A. El Jai, Sensor structures and regional detectability of parabolic distributes systems,*Sensors and Actuators* A, vol. 90, no. 3, pp. 163–171, **2001.**

[10] R. Al-Saphory and A. El Jai, Sensors characterizations for regional boundary detectability in distributed parameter systems, *Sensors and Actuators A*, vol. 94, no. 1-2, pp. 1–10, **2001.**

[11] R. Al-Saphory and M Al-Joubory and M Jasim,“Regional strategic sensors characterizations” *Journal of Mathematics and computational science*, preprint, **2013.**

[12] R. Al-Saphory,“strategic Sensors and regional exponential observability,” *International Scholarly Research Network*, vol. 2011, Article ID.673052, **2011**.

[13] E. Zerrik, L. Badraui, and A. El Jai, sensors and regional boundary state reconstruction of parabolic systems, *Sensors and Actuators A*, vol. 75, no. 7, pp. 102-117, **1999.**

[14] E.Zerrik, H. Bourray, and A. Boutoulout, Regional boundary observability: a numerical approach,*International Journal of Applied Mathematics and Computer Science*, vol. 12, no. 2, pp. 143–151, 2**002.**

[15] A. El Jai and H. Hamzaoui, Regional observation and sensors, *International Journal of Applied Mathematics and Computer Science*, vol. 19, no. 1, pp. 5 –14, **2009.**

[16] Dautray R.; Lions, J.L. Analyse mathématique et calcul numérique pour les sciences et lestechniques; série scientifique 8, Masson : Paris, 1984.

المستخلص:

الغرض من هذا البحث الموسوم (المجسات الستراتيجية مع القابلية على المشاهدة الاسية ضمن منطقة حدودية), هو توسيع مفهوم القابلية على المشاهدة الاسية المناطقية الى حالة المنطقة الحدودية سوية مع المجسات الستراتيجية. حيث سنأخذ نوع خاص من الانظمة وهي الانظمة التوزيعية وسنبرهن ان عدد ومواقع المجسات لها دور في وجود القابلية على المشاهدة الاسية ضمن تلك المنطقة الحدودية .